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Laminar natural convection about an isothermally heated sphere at small Grashof number

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The flow induced by gravity about a very small heated isothermal sphere introduced into a fluid in hydrostatic equilibrium is studied. The natural-convection flow is taken to be steady and laminar. The conditions under which the Boussinesq model is a good approximation to the full conservation laws are described. For a concentric finite cold outer sphere with radius, in ratio to the heated sphere radius, roughly less than the Grashof number to the minus one-half power, a recirculating flow occurs; fluid rises near the inner sphere and falls near the outer sphere. For a small heated sphere in an unbounded medium an ordinary perturbation expansion essentially in the Grashof number leads to unbounded velocities far from the sphere; this singularity is the natural-convection analogue of the Whitehead paradox arising in three-dimensional low-Reynolds-number forced-convection flows. Inner-and-outer matched asymptotic expansions reveal the importance of convective transport away from the sphere, although diffusive transport is dominant near the sphere. Approximate solution is given to the nonlinear outer equations, first by seeking a similarity solution (in paraboloidal co-ordinates) for a point heat source valid far from the point source, and then by linearization in the manner of Oseen. The Oseen solution is matched to the inner diffusive solution. Both outer solutions describe a paraboloidal wake above the sphere within which the enthalpy decays slowly relative to the rapid decay outside the wake. The updraft above the sphere is reduced from unbounded growth with distance from the sphere to constant magnitude by restoration of the convective accelerations. Finally, the role of vertical stratification of the ambient density in eventually stagnating updrafts predicted on the basis of a constant-density atmosphere is discussed.

1. Introduction

The flow (assumed steady and laminar) induced about a small heated sphere introduced into a fluid in hydrostatic equilibrium is examined in the Boussinesq approximation (Spiegel & Veronis 1960). This approximation treats the fluid as incompressible and of constant properties, except for the buoyancy force; in that term, for small deviations from hydrostatic equilibrium owing to nonuniform heating in a gravitational field, the deviation from ambient density may be linearly related to the deviation from ambient of the other thermodynamic variables describing the thermodynamic state of the fluid. The ambient density is here taken as effectively constant; that is, the distance over which the ambient density falls an appreciable fraction is taken to be much larger than other typical lengths characterizing the region of interest.

The treatment of finite geometries in low-Grashof-number flow has received relatively little attention. Nevertheless, the hot-sphere problem bears directly on droplet behaviour in a meteorological or liquid-rocket-engine context. Furthermore, the problem reveals a natural-convection analogue (Mahony 1956) of the Whitehead paradox of forced convection (Kaplun & Lagerstrom 1957; Proudman & Pearson 1957). The Whitehead paradox concerns the observation that, for unbounded steady low-Reynolds-number flow past a finite-size three-dimensional body, an asymptotic expansion in the Reynolds number based on the dominance of diffusive over convective transport everywhere in the flow field is not uniformly valid beyond the lowest-order (Stokes) solution.

The problem of a slightly heated sphere introduced into a hydrostatically stratified fluid is seemingly amenable to small-perturbation analysis. If one 'switches on' a gravitational field in an already developed flow of great extent parallel to the field, small-perturbation analysis is inappropriate.

2. The general mathematical formulation and special solution for a finite outer sphere

If gravity acts in the negative z-direction, and if constant Prandtl number and specific heats are adopted, then the non-dimensionalized boundary-value problem is

$$\nabla . (\rho \mathbf{q}) = 0, \tag{1}$$

$$G\rho[\nabla(\tfrac{1}{2}q^2) + (\nabla \times \mathbf{q}) \times \mathbf{q}] = -\nabla p + \nabla . \boldsymbol{\tau} - \rho \hat{z}, \tag{2}$$

$$G\rho \mathbf{q} . \nabla h = \frac{1}{Pr} \nabla^2 \left[\int_1^h \bar{\mu}(h_1)\, dh_1 \right] + (\gamma - 1) M^2 (\Phi + \mathbf{q} . \nabla p), \tag{3}$$

$$\frac{p}{\alpha} = \rho h, \quad \Phi = \operatorname{def} \mathbf{q} : \boldsymbol{\tau}, \quad \boldsymbol{\tau} = \lambda(\nabla . \mathbf{q})\mathbf{I} + \bar{\mu} \operatorname{def} \mathbf{q}, \tag{4}$$

$$r = 1: \qquad \mathbf{q} = 0, \quad h = 1 + \epsilon, \tag{5}$$

$$r = R: \qquad \mathbf{q} = 0, \quad h = 1. \tag{6}$$

A partial list of symbols and definitions is given at the end of the paper. Jacob (1949) termed G the gravity number and Eshghy & Morrison (1966) called M^2 the compressibility factor. The parameter α, related to lapse rate effects (stratification of the ambient density in the direction of the impressed gravitational body force), is $\gamma M^2/G$. The reference thermodynamic state is that which existed at the centre of the sphere prior to the introduction of the heated sphere. For simplicity only, the ambient enthalpy will be taken as constant. The parameter ϵ = (enthalpy of the heated sphere − ambient enthalpy)/ambient enthalpy. Equation (6) considers the heated sphere to lie within a concentric but finite outer spherical container. Later, the case $R \to \infty$ is considered, and (6) is relaxed to

$$r \to \infty: \qquad q_\theta, q_r \text{ bounded}; \quad h \to 1. \tag{7}$$

The next heat flux through any concentric spherical surface must equal the total heat flux at the sphere surface for a steady state to persist. Purported solutions violating this condition, even if only at $r \to \infty$ (Hossain 1966), are not satisfactory. It may be noted that the Grashof number does not in general appear; its constituent non-dimensional factors ϵ and G act independently (Jacob 1949). The solution possesses azimuthal symmetry; the initial ray of the polar angle θ is taken to be antiparallel to gravity.

For the case $$R = o[(\epsilon G)^{-\frac{1}{2}}], \quad G \leqslant O(1) \quad \text{and} \quad \epsilon \ll 1,$$

solution is sought in the form of a conventional perturbation expansion (Ostrach 1964); in this special case the Grashof number ϵG appears:

$$\rho = \rho_e + \epsilon\rho_1 + o(\epsilon), \tag{8}$$

$$h = 1 + \epsilon h_1 + o(\epsilon), \tag{9}$$

$$p = \alpha p_e + \epsilon p_1 + o(\epsilon), \tag{10}$$

$$\mathbf{q} = \mathbf{0} + \epsilon \mathbf{q}_1 + o(\epsilon), \tag{11}$$

$$\bar{\mu} = 1 + \epsilon\mu_1 + o(\epsilon), \tag{12}$$

$$\lambda = \lambda_\infty + \epsilon\lambda_1 + o(\epsilon), \tag{13}$$

where the viscosity coefficients are taken to be functions of the enthalpy only;

$$\alpha\nabla p_e = -\rho_e\hat{z}, \quad p_e = \rho_e \Rightarrow p_e = \rho_e = e^{-z/\alpha} \simeq 1 \quad \text{for} \quad \alpha \gg z, \tag{14}$$

$$\nabla \cdot \mathbf{q}_1 = 0, \quad \rho_1 = -h_1 \quad \text{for} \quad \epsilon \gg \alpha^{-1}, \tag{15}$$

$$\epsilon G[\nabla(\tfrac{1}{2}q_1^2) + (\nabla \times \mathbf{q}_1) \times \mathbf{q}_1] = -\nabla p_1 - \rho_1\hat{z} - \nabla \times (\nabla \times \mathbf{q}_1), \tag{16}$$

$$\epsilon G(\mathbf{q}_1 \cdot \nabla h_1) = \frac{1}{Pr}\nabla^2 h_1 \quad \text{for} \quad M^2 \ll 1. \tag{17}$$

Equations (15)–(17) represent the Boussinesq approximation (Landau & Lifshitz 1959); they have been applied to the present problem, without formal justification, by Mahony (1956).

Here $\epsilon G \ll O(1)$ is examined so the convective transport in (15) and (16) is neglected as a lowest-order approximation. It will now be shown that, despite implications to the contrary (Ostrach 1964), the resulting linear set of equations is often not adequate for treating small Grashof number flows. If

$$\mathbf{q}_1 = \nabla \times \left[\frac{\psi_1(r,\mu)}{r(1-\mu^2)^{\frac{1}{2}}}\hat{\phi}\right], \tag{18}$$

then $$\nabla^2 h_1 = D^4\psi_1 - 2Q_1(\mu)\left(r\frac{\partial h_1}{\partial r} - \mu\frac{\partial h_1}{\partial \mu}\right) = 0. \tag{19}$$

Hence, since $h_1(1) - 1 = h_1(R) = 0$,

$$\rho_1 = -h_1 = \frac{1}{R-1}\left(\frac{r-R}{r}\right), \tag{20}$$

$$\psi_1 = \left(A_1 r^4 + B_1 r^2 + C_1 r + D_1 r^{-1} + \frac{1}{4}\frac{R}{R-1}r^3\right)Q_1(\mu), \tag{21}$$

$$p_1 = -\left[\frac{r}{R-1} + 10A_1 r + C_1 r^{-2}\right] P_1(\mu), \tag{22}$$

where A_1, B_1, C_1 and D_1 are constants of integration assigned by the no-slip conditions

$$\psi_1(1) = \psi_1(R) = \frac{\partial \psi_1(1)}{\partial r} = \frac{\partial \psi_1(R)}{\partial r} = 0.$$

The algebraic details are omitted but, of course, calculation shows that the flow is such that fluid rises near the warmer inner sphere and falls near the colder outer sphere in a simple one-cell recirculation.

If q_h^* is the magnitude of the heat flux density to the fluid and k_∞^* is the ambient thermal conductivity, then the Nusselt number Nu is

$$Nu = \frac{q_h^*}{2\pi k_\infty^*(T_s^* - T_\infty^*)a^*} = -\frac{1}{\epsilon}\int_{-1}^{+1}\left(\mu\frac{\partial h}{\partial r}\right)_{r=1} d\mu = \frac{2R}{R-1} + o(1). \tag{23}$$

As $R \to \infty$, $|\mathbf{q}_1| \to \infty$; in fact, $|\mathbf{q}_1|$ grows linearly with r. (Despite this singular behaviour the Nusselt number remains bounded and approaches 2.) Whenever even a small temperature difference is maintained over a great expanse *parallel* to a gravitational field, large speeds may be induced. The expansions (8)–(13) are invalid far from the sphere because the sphere radius is no longer a typical length and diffusive transport no longer dominates convective transport. Substitution of the diffusive solution (20)–(22) into (16) and (17) reveals that, as $R \uparrow (\epsilon G)^{-\frac{1}{2}}$, the convective terms cannot be neglected. The expansions (8)–(13) comprise only the inner expansions of an inner-and-outer (matched asymptotic) expansion procedure. An outer expansion valid far from the sphere must be introduced; traditionally, at least far from the sphere, to lowest order in the outer field the sphere is reduced to a point source of momentum and energy.

3. The inner expansion for a sphere in an unbounded medium

The case $0 < \epsilon \ll G \ll 1$ is examined and the expansions (8), (9), (12) and (13) are again adopted for the enthalpy and density close to the sphere. However, the expansions for the pressure and velocity are generalized:

$$\mathbf{q} = \mathbf{0} + \left(\frac{\epsilon}{G}\right)^{\frac{1}{2}} \mathbf{q}_0 + \epsilon \mathbf{q}_1 + o(\epsilon), \tag{24}$$

$$p = p_e + \left(\frac{\epsilon}{G}\right)^{\frac{1}{2}} p_0 + \epsilon p_1 + o(\epsilon). \tag{25}$$

The motivations for such a generalization are many. First, just as the energy equation is diffusively dominated, so now is the momentum equation to lowest order beyond hydrostatic conditions. The lowest-order velocity is now $O[(\epsilon g^* a^*)^{\frac{1}{2}}]$ and arises from a buoyancy-driven outer flow shearing the fluid near the inner sphere; such a driving mechanism could not arise for $R = o[(\epsilon G)^{-\frac{1}{2}}]$. The particular choice $(\epsilon/G)^{\frac{1}{2}}$ permits the convective transport of momentum and energy, neglected in the equations for $\mathbf{q}_0$ and p_0, to serve as forcing functions in the equations for $\mathbf{q}_1$, p_1 and h_1—a commonly occurring feature of perturbation expansions.

Matching to the outer solution ultimately determines whether or not $\mathbf{q}_0$ and p_0 vanish, for, although it still holds that

$$\psi_0(1,\mu) = \frac{\partial\psi_0(1,\mu)}{\partial r} = \psi_1(1,\mu) = \frac{\partial\psi_1(1,\mu)}{\partial r} = h_1(1,\mu)-1 = 0, \tag{26}$$

from (5), the remaining constants of integration are assigned on the basis of compatibility with the outer expansion in domains of overlapping validity.

The governing equations for p_e and ρ_e remain those given in (14). But now

$$\nabla.\mathbf{q}_0 = \nabla p_0 + \nabla\times(\nabla\times\mathbf{q}_0) = 0, \tag{27}$$

$$\nabla.\mathbf{q}_1 = 0, \quad \rho_1 = -h_1, \tag{15}$$

$$-\nabla p_1 - \rho_1\hat{z} - \nabla\times(\nabla\times\mathbf{q}_1) = \nabla(\tfrac{1}{2}q_0^2) + (\nabla\times\mathbf{q}_0)\times\mathbf{q}_0, \tag{28}$$

$$\nabla^2 h_1 = 0. \tag{29}$$

It is anticipated in view of (20)–(22) that one may write

$$h_1 = [A_0 + (1-A_0)r^{-1}], \tag{30}$$

$$\psi_0 = B_1(r^2 - \tfrac{3}{2}r + \tfrac{1}{2}r^{-1})\,Q_1(\mu), \tag{31}$$

where the boundary conditions at $r = 1$ have been satisfied. Then

$$p_0 = -\frac{3B_1}{2r^2}P_1(\mu) + \text{const.} \tag{32}$$

The curl and divergence of (28) give

$$D^4\psi_1 = 2Q_1(\mu)\left(r\frac{\partial h_1}{\partial r} - \mu\frac{\partial h_1}{\partial\mu}\right) + \frac{1}{r^2}\left[\frac{\partial(\psi_0, D^2\psi_0)}{\partial(r,\mu)} + 2(D^2\psi_0)(L\psi_0)\right], \tag{33}$$

$$\nabla^2(p_1 + \tfrac{1}{2}q_0^2) = \hat{z}.\nabla h_1 - \nabla.[(\nabla\times\mathbf{q}_0)\times\mathbf{q}_0]. \tag{34}$$

Use of the expressions for h_1 and ψ_0 in these last two equations gives, in view of the no-slip boundary conditions,

$$\begin{aligned}\psi_1 = \sum_{\substack{n=0\\ n\neq 1,2}}^{\infty}\Bigg\{&A_n^*\left[r^{n+3} - \frac{2n+3}{2}r^{2-n} + \frac{2n+1}{2}r^{-n}\right]Q_n(\mu)\\ &+ B_n^*\left(r^{n+1} - \frac{2n+1}{2}r^{2-n} + \frac{2n-1}{2}r^{-n}\right)Q_n(\mu)\Bigg\} + \Bigg[A_1^*\left(r^4 - \frac{5}{2}r + \frac{3}{2r}\right)\\ &+ B_1^*\left(r^2 - \frac{3}{2}r + \frac{1}{2r}\right) + (1-A_0)\left(\frac{r^3}{4} - \frac{r}{2} + \frac{1}{4r}\right)\Bigg]Q_1(\mu) + \Bigg[A_2^*\left(r^5 - \frac{7}{2} + \frac{5}{2}r^{-2}\right)\\ &+ B_2^*\left(r^3 - \frac{5}{2} + \frac{3}{2}r^{-2}\right) + \frac{3}{8}B_1^2\left(\frac{1}{2r^2} + \frac{1}{2} + r^2 - \frac{3}{2}r - \frac{1}{2r}\right)\Bigg]Q_2(\mu),\end{aligned} \tag{35}$$

$$\begin{aligned}p_1 = -\tfrac{1}{2}q_0^2 + \sum_{n=0}^{\infty}\Bigg((a_n r^n + b_n r^{-n-1})P_n(\mu) + \frac{P_1(\mu)}{2}(1-A_0) + 3B_1^2\Bigg\{&\frac{1}{8}\left[\frac{1}{r^4} + \frac{1}{6r^6}\right]P_0(\mu)\\ &+ \left[-\frac{1}{6r} + \frac{1}{24r^4} + \frac{7}{16r^2}\right]P_2(\mu)\Bigg\}.\end{aligned} \tag{36}$$

4. The outer expansion

Far from the sphere the strained radial co-ordinate $\tilde{r}$ is introduced, where

$$\tilde{r} = (G\epsilon)^{\frac{1}{2}} r, \tag{37}$$

$$\rho(\tilde{r}, \mu, G, \epsilon) = \rho_e(z) + \epsilon^{\frac{3}{2}} G^{\frac{1}{2}} \Sigma(\tilde{r}, \mu) + \ldots, \tag{38}$$

$$h(\tilde{r}, \mu, G, \epsilon) = 1 + \epsilon^{\frac{3}{2}} G^{\frac{1}{2}} \Theta(\tilde{r}, \mu) + \ldots, \tag{39}$$

$$\mathbf{q}(\tilde{r}, \mu, G, \epsilon) = \epsilon^{\frac{1}{2}} G^{-\frac{1}{2}} \mathbf{Q}(\tilde{r}, \mu) + \ldots, \tag{40}$$

$$\bar{\mu}(\tilde{r}, \mu, G, \epsilon) = 1 + \epsilon^{\frac{3}{2}} G^{\frac{1}{2}} \tilde{\mu}_1(\tilde{r}, \mu) + \ldots, \tag{41}$$

$$\lambda(\tilde{r}, \mu, G, \epsilon) = \lambda_e + \epsilon^{\frac{3}{2}} G^{\frac{1}{2}} \tilde{\lambda}_1(\tilde{r}, \mu) + \ldots, \tag{42}$$

$$p(\tilde{r}, \mu, G, \epsilon) = \alpha p_e(z) + \epsilon P(\tilde{r}, \mu) + \ldots. \tag{43}$$

These scalings, except for those suggested for density and enthalpy, are exactly equivalent to those proposed by Mahony (1956). Substituting (38)–(43) into (1)–(4) gives (taking $\rho_e \simeq 1$)

$$\Sigma = -\Theta, \quad \tilde{\nabla} . \mathbf{Q} = 0, \tag{44}$$

$$\tilde{\nabla}(\tfrac{1}{2}\tilde{Q}^2) + (\tilde{\nabla} \times \mathbf{Q}) \times \mathbf{Q} = -\tilde{\nabla} P + \Theta \hat{z} - \tilde{\nabla} \times (\tilde{\nabla} \times \mathbf{Q}), \tag{45}$$

$$\mathbf{Q} . \tilde{\nabla} \Theta = \frac{1}{Pr} \tilde{\nabla}^2 \Theta. \tag{46}$$

Equation (7) requires that: $\Theta(\tilde{r} \to \infty, \mu) \to 0$, $\mathbf{Q}(\tilde{r} \to \infty, \mu)$ is bounded.

The use of asymptotic expansions has still left a lowest-order problem (above hydrostatic) that is formidably non-linear if one wishes an exact solution. First, a solution to the non-linear set valid for $\tilde{r} \gg 1$ will be attempted by similarity analysis. Such a solution cannot be meaningfully matched to the inner expansion since such a matching occurs in the limit $\tilde{r} \to 0$. Then, an approximate solution will be sought by linearizing the non-linear convective terms in the manner of Oseen; the motivation for so proceeding is furnished by properties of the similarity solution.

5. The point-source solution for the outer flow

Zel'dovich (1937) noted on dimensional grounds that, if one neglected axial diffusion of momentum and energy, (44)–(46) admitted of a similarity solution in the laminar axisymmetric case in terms of the cylindrical polar co-ordinate $\tilde{\rho}^2/\tilde{z}$ (see also Prandtl 1952). Here $\tilde{\rho}$ is the two-dimensional radial co-ordinate. Shuh (1948) was the first to state the formal boundary-value problem, to which Gutman (1949) found numerical solutions and Yih (1956) closed-form solutions for special values of the Prandtl number. Mahony (1956), Fujii (1963) and Brand & Lahey (1967) have repeated the derivation in recent years without adding any new findings. Although the planar case is of no direct relevance here, it may be noted that analogous results were found by the same group of authors to the two-dimensional case of a horizontal line source. Furthermore, Schmidt (1941) and Szablewski (1966) have derived similar results for point and line sources in turbulent fields in which exchange processes are modelled by simple mixing-length theories. Fendell & Smith (1967) later applied the similarity theory to the full equations (44)–(46) and showed that within a paraboloidal region in the far field above the body (i.e. for $\tilde{z} \gg 1$, $\tilde{z} \gg \tilde{\rho}$) the similarity theory was indeed a first approximation to the far-field solution. In this sense the point-source solution represents the lowest-order term in a co-ordinate expansion within the first non-trivial outer problem of the matched asymptotic parametric expansion. Clearly the matching of the inner and outer solutions involves the limit $\tilde{r} \to 0$, and thus the point-source solution is unsatisfactory for matching. However, the point-source solution does characterize the outer flow in a way that will be exploited later.

Here a solution, valid everywhere (except $\mu = -1$) for $\tilde{r} \gg 1$, will be sought for a point heat source at the origin of co-ordinates. Instead of the cylindrical polar co-ordinates adopted by all previous workers, use is here made of rotationally symmetric parabolic co-ordinates:

$$\tilde{z} + i\tilde{\rho} = \tfrac{1}{2}(\xi + i\eta)^2. \tag{47}$$

Equations (44)–(46) become ($\mathbf{Q} = Q_\xi \hat{\xi} + Q_\eta \hat{\eta}$):

$$\frac{\partial}{\partial \xi}[\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}} Q_\xi] + \frac{\partial}{\partial \eta}[\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}} Q_\eta] = 0, \tag{48}$$

$$Q_\xi \frac{\partial Q_\xi}{\partial \xi} + Q_\eta \frac{\partial Q_\xi}{\partial \eta} + Q_\eta(\eta Q_\xi - \xi Q_\eta) = -\frac{\partial P}{\partial \xi} + \xi\Theta + \frac{1}{\xi\eta}\frac{\partial}{\partial \eta}\left\{\frac{\xi\eta}{(\xi^2+\eta^2)^{\frac{3}{2}}}\left[(\xi^2+\eta^2)\left(\frac{\partial Q_\xi}{\partial \eta} - \frac{\partial Q_\eta}{\partial \xi}\right) + (\eta Q_\xi - \xi Q_\eta)\right]\right\}, \tag{49}$$

$$Q_\xi \frac{\partial Q_\eta}{\partial \xi} + Q_\eta \frac{\partial Q_\eta}{\partial \eta} + Q_\xi(\xi Q_\eta - \eta Q_\xi) = -\frac{\partial P}{\partial \eta} - \eta\Theta + \frac{1}{\xi\eta}\frac{\partial}{\partial \xi}\left\{\frac{\xi\eta}{(\xi^2+\eta^2)^{\frac{3}{2}}}\left[(\xi^2+\eta^2)\left(\frac{\partial Q_\eta}{\partial \xi} - \frac{\partial Q_\xi}{\partial \eta}\right) + (\xi Q_\eta - \eta Q_\xi)\right]\right\}, \tag{50}$$

$$Q_\xi \frac{\partial \Theta}{\partial \xi} + Q_\eta \frac{\partial \Theta}{\partial \eta} = \frac{1}{Pr(\xi^2+\eta^2)^{\frac{1}{2}}}\left(\frac{\partial^2\Theta}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \Theta}{\partial \xi} + \frac{\partial^2\Theta}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \Theta}{\partial \eta}\right). \tag{51}$$

The following similarity forms, motivated by Zel'dovich's original arguments, are adopted:

$$Q_\xi = \frac{1}{\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}}}\frac{\partial \psi}{\partial \eta}, \quad Q_\eta = \frac{-1}{\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}}}\frac{\partial \psi}{\partial \xi}, \tag{52}$$

where $\psi = (\frac{1}{2}\xi^2) f(\eta^2)$. Further,

$$\Theta = (\xi^2+\eta^2)^{-1}\bar{\theta}(\eta^2), \tag{53}$$

$$P = \bar{p}(\xi, \eta^2). \tag{54}$$

At this point is is quite useful to anticipate that to lowest order for $\tilde{r} \gg 1$ conduction in the ξ-direction is negligible compared with conduction in the η-direction. Then, combining continuity and energy conservation yields

$$\frac{\partial}{\partial \xi}[\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}} Q_\xi \Theta] + \frac{\partial}{\partial \eta}[\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}} Q_\eta \Theta] = \frac{1}{Pr}\xi\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \Theta}{\partial \eta}\right). \tag{55}$$

If, as $\eta \to \infty$, $\Theta = o(\eta^{-1})$ and $Q_\eta \Theta = o(\eta^{-2})$, then (since analytic behaviour of all dependent variables as $\eta \to 0$ is required) integration over all η yields

$$\frac{d}{d\xi}\int_0^\infty \xi\eta(\xi^2+\eta^2)^{\frac{1}{2}} Q_\xi \Theta\, d\eta = \frac{d}{d\xi}\int_0^\infty \frac{\partial\psi}{\partial\eta}\Theta\, d\eta = 0. \tag{56}$$

If $x = \eta^2$, then substitution of the similarity forms gives

$$[f(\eta) = F(x),\ \bar{\theta}(\eta) = \tau(x)]$$

$$\frac{d}{d\xi}\left(\frac{1}{2}\int_0^\infty \frac{\xi^2}{\xi^2+x} F'\tau\, dx\right) = 0. \tag{57}$$

If $\xi^2 \gg 1$,

$$\int_0^\infty F'\tau\, dx \doteqdot 2H, \tag{58}$$

where H is the net heat flux from the sphere to the fluid. The quantity H may be related to parameters arising in the boundary-value problem by use of (23):

$$H = 2\pi Nu/Pr. \tag{59}$$

Probably, as $R \to \infty$, $Nu \to 2$ to lowest order, but Nu will be left arbitrary for now.

Under the similarity forms (52)–(54), equations (48)–(51) become

$$\frac{\partial\bar{p}}{\partial x} = -\frac{1}{2R^2}\left[-\frac{F^2}{x} + x(F')^2 - 4F''\right] - \frac{1}{2R}\left[\tau + 4F'' + \frac{2FF'}{x} - \frac{F^2}{x^2} - (F')^2\right], \tag{60}$$

$$(4xF'')' + 2FF'' + \tau = \frac{R}{\xi}\frac{\partial\bar{p}}{\partial\xi} + \frac{1}{R}\left[x(F')^2 - \frac{F^2}{x} + 4xF''\right], \tag{61}$$

$$(2x\tau' + PrF\tau)' = \frac{1}{R}[4x\tau' + Pr\tau(xF' + F)], \tag{62}$$

where $R = \xi^2 + x = 2\tilde{r}$. For $\xi \gg 1$, the leading equations are

$$\frac{\partial\bar{p}}{\partial x} = 0, \tag{63}$$

$$2xF''' + FF'' = -\tfrac{1}{2}(4F'' + \tau), \tag{64}$$

$$2x\tau' + PrF\tau = 0. \tag{65}$$

The boundary condition $\Theta(\tilde{r} \to \infty, \mu) \to 0$ has been used to evaluate a constant of integration in (65). The other restraints on F and τ are

$$F(0) = 0, \tag{66}$$

$$\tau(0) = -4F''(0), \tag{67}$$

$$F'(x \to \infty) \to 0, \tag{68}$$

$$\int_0^\infty F'\tau\, dx = \frac{4\pi Nu}{Pr}, \tag{69}$$

where (66) and (67) assure analytic behaviour near the source at $x = 0$, and (68) assures that the velocity dies off far from the source (except directly above the source). Equation (69) expresses the invariance of the energy released per unit time, with distance from the source.

In general, (64) and (65), subject to (66)–(69), require numerical solution; however, Yih (1956) has given closed-form solutions for Prandtl numbers one and two. Specifically,

$$Pr = 1: \qquad F = \frac{6x}{B_1+x}, \quad \tau = -4F'' = \frac{48B_1}{(B_1+x)^3}, \quad B_1 = 3\left(\frac{2}{\pi Nu}\right)^{\frac{1}{2}}, \tag{70}$$

$$Pr = 2: \qquad F = \frac{4x}{B_2+x}, \quad \tau = \frac{32B_2^2}{(B_2+x)^4}, \quad B_2 = 8(5\pi Nu)^{-\frac{1}{2}}. \tag{71}$$

The streamlines describe a reverse-stagnation-like flow, i.e. an inflow towards the axis of symmetry and an updraft along the axis above the sphere. The isotherms are like tear-droplets stretching upward from the point source. It may readily be confirmed that the anticipations concerning the solution stated below (55) are fulfilled.

On the axis of symmetry above the point heat source (to which the hot sphere has been reduced by the co-ordinate stretching, at least for $\tilde{r} \gg 1$) the perturbation enthalpy Θ falls off only as $\tilde{r}^{-1}$; outside a parabolic wake above sphere Θ falls more rapidly ($\tilde{r}^{-4}$ for $Pr = 1$, $\tilde{r}^{-5}$ for $Pr = 2$). The velocity also displays a parabolic wake above the sphere. For $Pr = 1$, equations (52) and (70) give

$$Q_\xi = \left(\frac{1+\mu}{2}\right)^{\frac{1}{2}} \frac{6B_1}{[B_1+\tilde{r}(1-\mu)]^2}, \quad Q_\eta = -\left(\frac{1-\mu}{2}\right)^{\frac{1}{2}} \frac{6}{B_1+\tilde{r}(1-\mu)}. \tag{72}$$

For $\mu = 1 (\theta = 0)$, $Q_\eta = 0$ but $Q_\xi = 6/B_1$. The velocity does not go to zero, but rather remains invariant with height. Outside the wake there is a weakly decaying inflow that feeds the updraft:

$$Q_\eta \sim -\frac{1}{\tilde{r}(1-\mu)^{\frac{1}{2}}} \quad \text{as} \quad \tilde{r} \to \infty \quad (\mu \neq 1). \tag{73}$$

The task of finding a solution to the outer equations (44)–(46) valid as $\tilde{r} \to 0$ so that it may match to the inner solution [(30)–(32), (35) and (36)] remains. An approximate approach that models the convective terms adequately where they are important ($\tilde{r} \gg 1$), that retains the diffusive transport as $\tilde{r} \to 0$, and that yields tractable equations for all $\tilde{r}$ is required. For the analogous forced-convection problem Oseen linearized the non-linear terms about the prescribed uniform free-stream velocity; this empirical approximation was later given more basis for very small Reynolds numbers when matched inner-and-outer expansions yielded the Oseen equations as the correct far-field equations describing the first perturbation to the free-stream conditions. Weinbaum (1964) has successfully invoked the spirit of the Oseen linearization for a contained large-Grashof-number natural convection flow. Here the Oseen linearization will be applied to an unbounded low-Grashof-number flow by noting that, while no free-stream velocity is prescribed, an obvious choice for the velocity in the convective transport term naturally arises from the point-heat-source solution.

6. The Oseen linearization of the outer equations and matching of solutions

The Oseen linearization of (44)–(46) is (Carrier 1953)

$$\Sigma = -\Theta, \quad \tilde{\nabla}.\mathbf{Q} = 0, \tag{74}$$

$$\underline{\tilde{L}}\mathbf{Q} - \tilde{\nabla}P + \Theta\hat{z} = 0, \tag{75}$$

$$\underline{\tilde{L}}\Theta = 0, \tag{76}$$

where the Oseen operation $\underline{\tilde{L}}$ is

$$\underline{\tilde{L}} = \tilde{\nabla}^2 - C\frac{\partial}{\partial\tilde{z}}$$

and C is the Oseen constant, selected on the basis of the point-heat-source similarity solution. For the case $Pr = 1$, which has implicitly been adopted in writing (76), $C = 6/B_1$; for other Prandtl numbers, similar results can be derived. The outer boundary conditions stated after (46) for $\tilde{r}\to\infty$ still hold.

It is convenient to restate the problem posed in (74)–(76) in terms of two functions H and J where

$$\tilde{\nabla}^2 J = \Theta, \quad \underline{\tilde{L}}H = \Theta, \quad \underline{\tilde{L}}\Theta = 0. \tag{77}$$

Then
$$P = -C^{-1}\underline{\tilde{L}}(J-H) = \frac{\partial J}{\partial\tilde{z}}, \quad \mathbf{Q} = -H\hat{z} + C^{-1}\tilde{\nabla}(H-J). \tag{78}$$

The solution to the last of (77) is

$$\Theta = G(\tilde{r},\mu)\exp(\tfrac{1}{2}C\tilde{r}\mu), \tag{79}$$

$$G = \left(\frac{\pi}{C\tilde{r}}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} C_k K_{k+\frac{1}{2}}(\tfrac{1}{2}C\tilde{r})\,P_k(\mu), \tag{80}$$

where $K_{k+\frac{1}{2}}(\frac{1}{2}C\tilde{r})$ is the half-order modified Bessel function bounded at infinity:

$$K_{k+\frac{1}{2}}(\tfrac{1}{2}C\tilde{r}) = \left(\frac{\pi}{C\tilde{r}}\right)^{\frac{1}{2}} e^{-\frac{1}{2}C\tilde{r}} \sum_{m=0}^{k} \frac{(k+m)!}{(k-m)!\,m!\,(C\tilde{r})^m}. \tag{81}$$

It is convenient to match the inner and outer representations of the enthalpy h at this stage. Matching is carried out according to the requirement that the outer expansion to $O[f_n(\epsilon, G)]$ of the inner expansion to $O[g_n(\epsilon, G)]$ must be equal to the inner expansion to $O[g_n(\epsilon, G)]$ of the outer expansion to $O[f_n(\epsilon, G)]$. From (9), (29), (30), (39), (79) and (80) it is readily seen that $A_0 = 0$, $C_0 = C/\pi$ and $C_n = 0$ for $n > 0$. This result confirms anticipations implicit in writing (30) and gives

$$\Theta = \frac{\exp[+\frac{1}{2}C\tilde{r}(\mu-1]}{\tilde{r}}. \tag{82}$$

The outer solution for the enthalpy given by (39) and (82) is in fact uniformly valid through $O(\epsilon)$ for all $r \geqslant 1$. Equation (82) describes a parabolic wake above the sphere; on the vertical axis the perturbation enthalpy decays at a rate inversely proportional to $\tilde{r}$ as $\tilde{r}\to\infty$ (this agrees with the point-source result). Outside the wake the enthalpy decays exponentially in the Oseen linearization, although it decays only algebraically in the point-source solution. The generalization of the inner and outer solutions and matching procedures for a non-isothermally heated sphere is evident. Since near the sphere $h = 1+\epsilon r^{-1}+o(\epsilon)$, it is confirmed from (23) that to lowest order the Nusselt number $Nu = 2$, independent of the Prandtl number. An analogous result holds in low-Reynolds-number low-Mach-number forced-convection flow past an isothermal sphere.

Substitution of (82) in the middle equation of (77) yields

$$H = \left(\frac{\pi}{C\tilde{r}}\right)^{\frac{1}{2}} e^{\frac{1}{2}C\tilde{r}\mu} \sum_{k=0}^{\infty} D_k K_{k+\frac{1}{2}}(\tfrac{1}{2}C\tilde{r}) P_k(\mu) - \frac{1}{C} e^{\frac{1}{2}C\tilde{r}(\mu-1)}. \tag{83}$$

Substitution of (82) in the first equation of (77) yields

$$J = \sum_{k=0}^{\infty} (E_k \tilde{r}^k + F_k \tilde{r}^{-k-1}) P_k(\mu) - \frac{1}{C}\{E_i[-\tfrac{1}{2}C\tilde{r}(1-\mu)] - \ln[\tfrac{1}{2}C\tilde{r}(1-\mu)]\}, \tag{84}$$

where D_k, E_k and F_k are constants of integration; boundedness criteria have rejected the other modified Bessel function as a complementary function for H; and $E_i(-x)$ is the well-tabulated exponential integral function

$$E_i(-x) = -\int_x^{\infty} \frac{e^{-t}}{t}\, dt. \tag{85}$$

If $Q_r(\tilde{r}\to\infty, \mu)\to 0$, except possibly for $\mu = 1$, then $E_k = 0$ for all $k > 0$.

The Oseen linearization gives for large $\tilde{r}$

$$H \sim O\{-\exp[\tfrac{1}{2}C\tilde{r}(\mu-1)]\}, \tag{86}$$

$$J \sim \begin{cases} O\{+\ln[\tfrac{1}{2}C\tilde{r}(1-\mu)]\} & (\mu \neq 1), \\ O(1) & (\mu = 1). \end{cases} \tag{87}$$

Equations (86) and (87) predict a radial inflow and axial updraft, roughly the reverse of a classical potential stagnation-point flow. The axial velocity goes to zero algebraically on the axis of symmetry far below the sphere and goes to a constant, finite updraft on the axis far above the sphere. The radial influx (radial in a cylindrical sense) falls off inversely with distance from the axis of symmetry, far from the axis of symmetry; the radial influx is greater above the sphere than below it. The point-source solution developed earlier agrees in essence with these results.

The matching of velocities is readily carried out. From (37), (78), (83) and (84) as $\tilde{r}\to 0$ the outer expansion behaves as

$$q_r \sim \left(\frac{\epsilon}{G}\right)^{\frac{1}{2}} \frac{P_1(\mu)}{C} + (\epsilon)\,\tfrac{1}{4}r[-P_1(\mu)+P_2(\mu)] + \ldots, \tag{88}$$

$$q_\theta \sim -\left(\frac{\epsilon}{G}\right)^{\frac{1}{2}} \frac{(1-\mu^2)^{\frac{1}{2}}}{C} - \epsilon(\tfrac{3}{8}r)(\mu-1)(1-\mu^2)^{\frac{1}{2}} + \ldots, \tag{89}$$

where $D_k = F_k = 0$ for all $k \geqslant 0$ to permit matching to the inner solution, which can be found from (18), (24), (31) and (35). Since the inner solution is

$$q_r = -\left(\frac{\epsilon}{G}\right)^{\frac{1}{2}} B_1 \left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right) P_1(\mu) + o\left(\frac{\epsilon}{G}\right)^{\frac{1}{2}}, \tag{90}$$

$$q_\theta = \left(\frac{\epsilon}{G}\right)^{\frac{1}{2}} B_1 \left(1 - \frac{3}{4r} - \frac{1}{4r^3}\right)(1-\mu^2)^{\frac{1}{2}} + o\left(\frac{\epsilon}{G}\right)^{\frac{1}{2}}, \tag{91}$$

$B_1 = -C^{-1}$ for compatibility of inner and outer expansions. Consideration of the $O(\epsilon)$ inner terms for the velocity similarly requires, for compatibility with (88) and (89), that $A_n^* = B_n^* = 0$ for all $n \geqslant 0$ except $B_2^* = -\frac{1}{4}$ and both B_0^* and B_1^* are unassigned to this order.

From (78), (83) and (84)

$$P = -\frac{1}{C\tilde{r}}[1 - \exp\{\tfrac{1}{2}C\tilde{r}(\mu - 1)\}]. \tag{92}$$

From (43) and (92) the outer expansion as $\tilde{r} \to 0$ behaves as

$$p \sim \alpha p_e + \tfrac{1}{2}\epsilon[-P_0(\mu) + P_1(\mu)] + \dots. \tag{93}$$

The inner expansion for the pressure can be found from (25), (32) and (36). Compatibility requires that $a_n = 0$ for all $n \geqslant 0$ except $a_0 = 2^{-1}(C^{-2} - 1)$. Members of the set b_n are unassigned to this order, although some may be found to vanish by use of (28). However, the matching of the inner and outer pressure representations is established and the demonstration of the compatibility of the expansions completed.

7. The lapse-rate effect

The existence of a constant updraft above the sphere arises from adopting a constant-density ambient state; it would be observed for a heated sphere introduced into a not-too-tall liquid column. Gases are more compressible and, at least by the time one studies heights z of the order of α, which is the ambient density decay rate for an isothermal atmosphere, the vertical stratification of the ambient gas must be considered. The outer equations of state and of continuity may need modification but the basic term that accounts for stratification is the $\mathbf{q} \cdot \nabla p_e$ expression in the energy conservation relation. The reduction of ambient density with height eventually eliminates the density discrepancy of rising gas heated by the sphere. This 'energy barrier' stagnates and spreads out an updraft in the atmosphere, just as the outer sphere stagnates and spreads out an updraft in the concentric-sphere geometry examined earlier in § 2. The turbulent entrainment of cooler ambient air, also omitted in the current theory, hastens the deceleration.

Although many mathematical details of the inner-and-outer matching procedure have yet to be rigorously carried out for the constant-density problem, it would seem that the most important physical understanding would now be derived from a quantitative study of a buoyancy-driven column in a spatial domain tall enough so that vertical stratification becomes an important effect.

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Partial list of symbols

a^* radius of heated sphere

D^2 differential operator $\dfrac{\partial^2}{\partial r^2}+\dfrac{1-\mu^2}{r^2}\dfrac{\partial^2}{\partial \mu^2}$

G gravity number, $\rho_0^* U^* a^*/\overline{\mu}_0^*$

g^* magnitude of gravitational acceleration

h_0^* the uniform enthalpy of the hydrostatic atmosphere

h enthalpy divided by h_0^*

L differential operator $\dfrac{\mu}{1-\mu^2}\dfrac{\partial}{\partial r}+\dfrac{1}{r}\dfrac{\partial}{\partial \mu}$

M^2 compressibility factor $(U^*)^2/(\gamma-1)\,h_0^*$

p pressure divided by $\overline{\mu}_0^* U^*/a^*$

p_e pressure for hydrostatic equilibrium divided by p_0^*

p_0^* pressure at origin of co-ordinates prior to introduction of sphere

$\mathbf{q}$ velocity divided by U^*

Q_n Gegenbauer polynomial $\int_{-1}^{\mu} P_n(\mu_1)\,d\mu_1$

R outer sphere radius divided by a^*; also, $\xi^2+\eta^2 = 2\tilde{r}$

r radial co-ordinate divided by a^*

U^* quantity with the dimensions of a speed $\rho_0^* g^* (a^*)^2/\overline{\mu}_0^*$

α ambient density decay rate $p_0^*/(\overline{\mu}_0^* U^*/a^*)$

ϵ normalized temperature difference $(h^*_{\text{sphere}}-h_0^*)/h_0^*$

λ bulk viscosity coefficient divided by $\overline{\mu}_0^*$

μ $\cos\theta$ where θ is the spherical polar angle measured from a ray antiparallel to gravity

$\overline{\mu}$ shear viscosity coefficient divided by $\overline{\mu}_0^*$

ρ_0^* density at origin of co-ordinates prior to introduction of the sphere

ρ density divided by ρ_0^*

ρ_e density for hydrostatic equilibrium divided by ρ_0^*

ϕ azimuthal angle in spherical polar co-ordinates

Subscripts and superscripts

e pertaining to hydrostatic equilibrium

$*$ dimensional quantity

$\sim$ a scaled variable of the outer expansion

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